

A Value Distribution Result and Some Normality Criteria using Partial Sharing of Small Functions

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Abstract

In this paper, we first generalize a value distribution result of Lahiri and Dewan [4] and as an application of this result we prove a normality criterion using partial sharing of small functions. Further, in sequel normality criteria of Hu and Meng [3] and Ding, Ding and Yuan [1] are improved and generalized when the domain $D := \{z : |z| < R, 0 < R \leq \infty\}$.

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AMS subject classification: 30D35, 30D45

1 Introduction and Main Results

We assume that the reader is familiar with the theory of normal families of meromorphic functions on a domain $D \subseteq \mathbb{C}$, one may refer to [6].

The idea of sharing of values was introduced in the study of normality of families of meromorphic functions, for the first time, by W. Schwick [7] in 1989. Two non-constant meromorphic functions f and g are said to share a value $\omega \in \mathbb{C}$ IM (Ignoring multiplicities) if f and g have the same ω -points counted with ignoring multiplicities. If multiplicities of ω -points of f and g are counted, then f and g are said to share the value ω CM. For deeper insight into the sharing of values by meromorphic functions, one may refer to [10].

In this paper all meromorphic functions are considered on $D := \{z : |z| < R, 0 < R \leq \infty\}$ excepting Theorem A and Theorem 1.1, where the domain is the whole complex plane. A meromorphic function $\omega(z)$ is said to be a *small function* of a meromorphic function $f(z)$ if $T(r, \omega) = o(T(r, f))$ as $r \rightarrow R$. Further, we say that a meromorphic function f share a small function ω *partially* with a meromorphic function g if

$$\overline{E}(\omega, f) = \{z \in \mathbb{C} : f(z) - \omega(z) = 0\} \subseteq \overline{E}(\omega, g) = \{z \in \mathbb{C} : g(z) - \omega(z) = 0\},$$

where $\overline{E}(\omega, \phi)$ denotes the set of zeros of $\phi - \omega$ counted with ignoring multiplicities.

The function of the form $M[f] = f^{n_0}(f')^{n_1} \dots (f^{(k)})^{n_k}$ is called a *differential monomial* of f of *degree* $d = n_0 + n_1 + \dots + n_k$, where n_0, n_1, \dots, n_k are non-negative integers.

In the present discussion, we have used the idea of partial sharing of small functions in the study of normality of families of meromorphic functions. One can verify that a good amount of results on normal families proved by using the sharing of values can be proved under weaker hypothesis of partial sharing of values or small functions.

Lahiri and Dewan [4] proved the following result:

Theorem A Let f be a transcendental meromorphic function and $F = (f)^{n_0}(f^{(k)})^{n_1}$, where $n_0(\geq 2), n_1$ and k are positive integers such that $n_0(n_0 - 1) + (1 + k)(n_0 n_1 - n_0 - n_1) > 0$. Then

$$\left[1 - \frac{1 + k}{n_0 + k} - \frac{n_0(1 + k)}{(n_0 + k)\{n_0 + (1 + k)n_1\}}\right] T(r, F) \leq \overline{N}\left(r, \frac{1}{F - \omega}\right) + S(r, F)$$

for any small function $\omega(\neq 0, \infty)$ of f .

This is natural to ask whether Theorem A remains valid for a general class of monomials. In this direction, we have proved that it does hold for a larger class of monomials. Precisely, we have

Theorem 1.1. *Let f be a transcendental meromorphic function. Let*

$$F = f^{n_0}(f')^{n_1} \dots (f^{(k)})^{n_k}, \quad (1.1)$$

where k, n_0, n_1, \dots, n_k are non-negative integers with $k \geq 1, n_0 \geq 2$ and $n_k \geq 1$ such that

$$n_0(n_0 - 1) + \sum_{j=1}^k (j+1)(n_0 n_j - n_j - n_0) + (k-1)n_0 > 0. \quad (1.2)$$

Then

$$\begin{aligned} & \left[1 - \frac{1 + \frac{k(k+1)}{2}}{n_0 + \frac{k(k+1)}{2}} - \frac{n_0(1 + \frac{k(k+1)}{2})}{\{n_0 + \frac{k(k+1)}{2}\} \{n_0 + \sum_{j=1}^k (j+1)n_j\}} + o(1) \right] T(r, F) \\ & \leq \overline{N} \left(r, \frac{1}{F - \omega} \right) + S(r, F) \end{aligned} \quad (1.3)$$

for any small function $\omega (\neq 0, \infty)$ of f .

Note: When f has no poles then Theorem 1.1 holds without the condition (1.2).

As an application of Theorem 1.1, we prove a normality criterion using the idea of partial sharing of small functions.

Theorem 1.2. *Let \mathcal{F} be a family of meromorphic functions such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least $k \geq 2$. Let n_0, n_1, \dots, n_k be non-negative integers with $n_0 \geq 2, n_k \geq 1$ such that*

$$n_0(n_0 - 1) + \sum_{j=1}^k (j+1)(n_0 n_j - n_0 - n_j) + (k-1)n_0 > 0$$

Let $\omega(z)$ be a small function of each $f \in \mathcal{F}$ having no zeros and poles at the origin. If there exists $\tilde{f} \in \mathcal{F}$ such that $M[f]$ share ω partially with $M[\tilde{f}]$, for every $f \in \mathcal{F}$, then \mathcal{F} is a normal family.

Further, one can see that Theorem 4.1 of Hu and Meng [3] may be generalized to a class of monomials as

Theorem 1.3. *Let $k \in \mathbb{N}$ and \mathcal{F} be a family of non-constant meromorphic functions such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least k . Let n_0, n_1, \dots, n_k be non-negative integers with $n_0 \geq 2, n_k \geq 1$ such that*

$$n_0(n_0 - 1) + \sum_{j=1}^k (j+1)(n_0 n_j - n_0 - n_j) + (k-1)n_0 > 0.$$

Let $\omega(z)$ be a small function of each $f \in \mathcal{F}$ having no zeros and poles at the origin. If, for each $f \in \mathcal{F}$, $(M[f] - \omega)(z) = 0$ implies $|f^{(k)}(z)| \leq A$, for some $A > 0$, then \mathcal{F} is a normal family.

2 Proof of Main Results

Proof of Theorem 1.1: Since (see [8])

$$T(r, f) + S(r, f) \leq CT(r, F) + S(r, F)$$

and

$$T(r, F) \leq \left[n_0 + \sum_{j=1}^k (j+1)n_j \right] T(r, f) + S(r, f),$$

where C is a constant, it follows that $T(r, \omega) = S(r, F)$ as $r \rightarrow \infty$. Precisely, ω is a small function of f iff ω is a small function of F .

Now, by Second Fundamental Theorem of Nevanlinna for three small functions (see [2] pp. 47), we have

$$[1 + o(1)]T(r, F) \leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \omega}\right) + S(r, F). \quad (2.1)$$

Next, we have

$$\begin{aligned} \overline{N}\left(r, \frac{1}{F}\right) &\leq \overline{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^k \overline{N}_0\left(r, \frac{1}{f^{(j)}}\right) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^k j \left[\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) \right] + S(r, f) \\ &= \overline{N}\left(r, \frac{1}{f}\right) + \frac{k(k+1)}{2} \left[\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) \right] + S(r, f), \end{aligned}$$

where $\overline{N}_0\left(r, \frac{1}{f^{(j)}}\right)$ is the number of those zeros of $f^{(j)}$ in $|z| \leq r$ which are not the zeros of f .

That is,

$$\overline{N}\left(r, \frac{1}{F}\right) \leq \left[1 + \frac{k(k+1)}{2} \right] \overline{N}\left(r, \frac{1}{f}\right) + \frac{k(k+1)}{2} \overline{N}(r, f) + S(r, f). \quad (2.2)$$

Also, we can see that

$$N\left(r, \frac{1}{F}\right) - \overline{N}\left(r, \frac{1}{F}\right) \geq \left[(k+1)n_0 + \sum_{j=1}^k n_j - 1 \right] \overline{N}_{(k+1)}\left(r, \frac{1}{f}\right) + (n_0 - 1) \overline{N}_k\left(r, \frac{1}{f}\right), \quad (2.3)$$

where $\overline{N}_{(k+1)}\left(r, \frac{1}{f}\right)$ and $\overline{N}_k\left(r, \frac{1}{f}\right)$ are the counting functions ignoring multiplicities of those zeros of f whose multiplicity is $\geq k+1$ and $\leq k$ respectively.

Now from (2.2) and (2.3), we get

$$\begin{aligned}\overline{N}(r, \frac{1}{F}) &\leq \left[1 + \frac{k(k+1)}{2}\right] \overline{N}_{(k+1)}(r, \frac{1}{f}) \\ &\quad + \frac{\left[1 + \frac{k(k+1)}{2}\right]}{n_0 - 1} \left[N(r, \frac{1}{F}) - \overline{N}(r, \frac{1}{F}) - \left((k+1)n_0 + \sum_{j=1}^k n_j - 1 \right) \overline{N}_{(k+1)}(r, \frac{1}{f}) \right] \\ &\quad + \frac{k(k+1)}{2} \overline{N}(r, f) + S(r, f).\end{aligned}$$

That is,

$$\begin{aligned}\left[1 + \frac{\left(1 + \frac{k(k+1)}{2}\right)}{n_0 - 1}\right] \overline{N}(r, \frac{1}{F}) &\leq \left(1 + \frac{k(k+1)}{2}\right) \left(1 - \frac{(k+1)n_0 + \sum_{j=1}^k n_j - 1}{n_0 - 1}\right) \overline{N}_{(k+1)}(r, \frac{1}{f}) \\ &\quad + \frac{1 + \frac{k(k+1)}{2}}{n_0 - 1} N(r, \frac{1}{F}) + \frac{k(k+1)}{2} \overline{N}(r, f) + S(r, f).\end{aligned}$$

Since $\overline{N}(r, f) = \overline{N}(r, F)$ and $S(r, f) = S(r, F)$, we have

$$\begin{aligned}\overline{N}(r, \frac{1}{F}) &\leq \frac{1 + \frac{k(k+1)}{2}}{n_0 + \frac{k(k+1)}{2}} N(r, \frac{1}{F}) + \frac{\left(\frac{k(k+1)}{2}\right)(n_0 - 1)}{n_0 + \frac{k(k+1)}{2}} \overline{N}(r, f) + S(r, f) \\ &= \frac{1 + \frac{k(k+1)}{2}}{n_0 + \frac{k(k+1)}{2}} N(r, \frac{1}{F}) + \frac{\left(\frac{k(k+1)}{2}\right)(n_0 - 1)}{n_0 + \frac{k(k+1)}{2}} \overline{N}(r, F) + S(r, F).\end{aligned}$$

Therefore, (2.1) yields

$$[1+o(1)]T(r, F) \leq \overline{N}\left(r, \frac{1}{F-\omega}\right) + \frac{1 + \frac{k(k+1)}{2}}{n_0 + \frac{k(k+1)}{2}} N(r, \frac{1}{F}) + \frac{n_0(1 + \frac{k(k+1)}{2})}{n_0 + \frac{k(k+1)}{2}} \overline{N}(r, F) + S(r, F). \quad (2.4)$$

Also, if f has a pole of multiplicity p , then F has a pole of multiplicity

$$n_0 p + n_1(p+1) + \cdots + n_k(p+k) \geq n_0 + 2n_1 + \cdots + (k+1)n_k = n_0 + \sum_{j=1}^k (j+1)n_j$$

and therefore,

$$N(r, F) \geq \left[n_0 + \sum_{j=1}^k (j+1)n_j \right] \overline{N}(r, F). \quad (2.5)$$

Finally, from (2.4) and (2.5), we find that

$$\begin{aligned}
[1 + o(1)]T(r, F) &\leq \overline{N}\left(r, \frac{1}{F - \omega}\right) + \frac{1 + \frac{k(k+1)}{2}}{n_0 + \frac{k(k+1)}{2}} N\left(r, \frac{1}{F}\right) \\
&\quad + \frac{n_0(1 + \frac{k(k+1)}{2})}{(n_0 + \frac{k(k+1)}{2})(n_0 + \sum_{j=1}^k (j+1)n_j)} N(r, F) + S(r, F).
\end{aligned}$$

That is,

$$\begin{aligned}
&\left[1 - \frac{1 + \frac{k(k+1)}{2}}{n_0 + \frac{k(k+1)}{2}} - \frac{n_0(1 + \frac{k(k+1)}{2})}{(n_0 + \frac{k(k+1)}{2})(n_0 + \sum_{j=1}^k (j+1)n_j)} + o(1)\right] T(r, F) \\
&\leq \overline{N}\left(r, \frac{1}{F - \omega}\right) + S(r, F).
\end{aligned}$$

□

For the proof of Theorem 1.2, besides Theorem 1.1, we also need the following lemma which is a straight forward generalization of *Lemma 3* in [1].

Lemma 2.1. *Let f be a non-constant rational function with only zeros of multiplicity at least k , where $k \geq 2$. Let $n_0, n_1, n_2, \dots, n_k$ be non-negative integers with $n_0 \geq 2$ and $n_k \geq 1$. Let $\omega \neq 0$ be a finite complex number. Then $M[f] - \omega$ has at least two distinct zeros.*

Proof of Theorem 1.2: Since normality is a local property, we may assume that $D = \mathbb{D}$. Suppose \mathcal{F} is not normal in \mathbb{D} . In particular, suppose that \mathcal{F} is not normal at $z = 0$. Then, by Zalcman's lemma (see [11]), there exist a sequence $\{f_n\}$ of functions in \mathcal{F} , a sequence $\{z_n\}$ of complex numbers in \mathbb{D} with $z_n \rightarrow 0$ as $n \rightarrow \infty$, and a sequence $\{\rho_n\}$ of positive real numbers with $\rho_n \rightarrow 0$ as $n \rightarrow \infty$ such that the sequence $\{g_n\}$ defined by

$$g_n(z) = \rho_n^{-\alpha} f_n(z_n + \rho_n z); 0 \leq \alpha < k,$$

converges locally uniformly to a non-constant meromorphic function $g(z)$ in \mathbb{C} with respect to the spherical metric. Moreover, $g(z)$ is of order at most 2. By Hurwitz's theorem, the zeros of $g(z)$ have multiplicity at least k .

Let $\alpha = \frac{\sum_{j=1}^k j n_j}{\sum_{j=0}^k n_j} < k$. Then

$$\begin{aligned}
M[g_n](z) &= (g_n(z))^{n_0} (g'_n(z))^{n_1} \dots \left(g_n^{(k)}(z)\right)^{n_k} \\
&= \rho_n^{-\alpha n_0} (f_n(z_n + \rho_n z))^{n_0} \rho_n^{-\alpha n_1 + n_1} (f'_n(z_n + \rho_n z))^{n_1} \dots \rho_n^{-\alpha n_k + k n_k} \left(f_n^{(k)}(z_n + \rho_n z)\right)^{n_k} \\
&= \rho_n^{-\alpha \sum_{j=0}^k n_j + \sum_{j=1}^k j n_j} (f_n(z_n + \rho_n z))^{n_0} (f'_n(z_n + \rho_n z))^{n_1} \dots \left(f_n^{(k)}(z_n + \rho_n z)\right)^{n_k} \\
&= M[f_n](z_n + \rho_n z).
\end{aligned}$$

On every compact subset of \mathbb{C} that contains no poles of g , we have

$$M[f_n](z_n + \rho_n z) - \omega(z_n + \rho_n z) = M[g_n](z) - \omega(z_n + \rho_n z) \rightarrow M[g](z) - \omega_0$$

spherically uniformly, where $\omega_0 = \omega(0)$.

Since g is a non-constant meromorphic function of order at most 2 and $\omega_0 \neq 0, \infty$, it immediately follows that $M[g] \not\equiv \omega_0$. Using Theorem 1.1 and Lemma 2.1, $M[g] - \omega_0$ has at least two distinct zeros, say, w_0 and v_0 . Choose $r > 0$ such that the open disks $D(w_0, r) = \{z : |z - w_0| < r\}$ and $D(v_0, r) = \{z : |z - v_0| < r\}$ are disjoint and their union contains no zeros of $M[g] - \omega_0$ different from w_0 and v_0 respectively. Then, by Hurwitz's theorem, we see that for sufficiently large n , there exist points $w_n \in D(w_0, r)$ and $v_n \in D(v_0, r)$ such that

$$(M[f_n] - \omega)(z_n + \rho_n w_n) = 0,$$

and

$$(M[f_n] - \omega)(z_n + \rho_n v_n) = 0.$$

Since by hypothesis, $M[f_n]$ share ω partially with $M[\tilde{f}]$, for every n , it follows that

$$(M[\tilde{f}] - \omega)(z_n + \rho_n w_n) = 0,$$

and

$$(M[\tilde{f}] - \omega)(z_n + \rho_n v_n) = 0.$$

By letting $n \rightarrow \infty$, and noting that $z_n + \rho_n w_n \rightarrow 0$, $z_n + \rho_n v_n \rightarrow 0$, we find that

$$(M[\tilde{f}] - \omega)(0) = 0.$$

Since the zeros of $M[\tilde{f}] - \omega$ have no accumulation point, $z_n + \rho_n w_n = 0$ and $z_n + \rho_n v_n = 0$ for sufficiently large n . That is, $D(w_0, r) \cap D(v_0, r) \neq \emptyset$, a contradiction. \square

Proof of Theorem 1.3: As established in the proof of Theorem 1.2, we similarly find that $M[g] \not\equiv \omega_0$. By Theorem 1.1 and Lemma 2.6 in [12], $M[g] - \omega_0$ has at least one zero w_0 , say. By Hurwitz's Theorem, there is a sequence of complex numbers $\{w_n\}$ such that $w_n \rightarrow w_0$ as $n \rightarrow \infty$, and

$$(M[f_n] - \omega)(z_n + \rho_n w_n) = 0$$

Again, since $k > \alpha$,

$$\begin{aligned} |g_n^{(k)}(w_n)| &= \rho_n^{k-\alpha} |f_n^{(k)}(z_n + \rho_n w_n)| \\ &\leq \rho_n^{(k-\alpha)} A \\ &= A \rho_n^{k - \frac{\sum_{j=1}^k j n_j}{\sum_{j=0}^k n_j}} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore, $g^{(k)}(w_0) = \lim_{n \rightarrow \infty} g_n^{(k)}(w_n) = 0$
 $\Rightarrow M[g](w_0) = 0 \neq \omega_0$, which is a contradiction. \square

3 Conclusions

Though our results do generalize and improve the results of Hu and Meng [3] and Ding, Ding and Yuan[1] when the domain D is $\{z : |z| < R, 0, R \leq \infty\}$, there seems no way of proving our results on arbitrary domain since the idea of small function on arbitrary domain is not available, as far as we know. However, by making certain modifications in the proofs of results of Hu and Meng[3] and Ding, Ding and Yuan[1], one can easily extend and improve these results on arbitrary domain with shared value being a non-zero complex value. Precisely, one obtains,

Theorem 3.1. *Let \mathcal{F} be a family of non-constant meromorphic functions on a domain D with all zeros of each $f \in \mathcal{F}$ having multiplicity at least k , where $k \geq 2$. Let $\omega \neq 0$ be a finite complex number and n_0, n_1, \dots, n_k be non-negative integers with $n_0 \geq 2$ and $n_1 + n_2 \dots + n_k \geq 1$. If there exists $\tilde{f} \in \mathcal{F}$ such that $M[\tilde{f}]$ share ω partially with $M[f]$ for every $f \in \mathcal{F}$, then \mathcal{F} is normal on D .*

The condition that f has only zeros of multiplicity atleast k in Theorem 3.1 is sharp. For example, consider the open unit disk \mathbb{D} , an integer $k \geq 2$, a non-zero complex number ω and the family

$$\mathcal{F} = \{f_m(z) = mz^{k-1}, m = 1, 2, 3, \dots\}$$

Obviously, each $f_m \in \mathcal{F}$ has only a zero of multiplicity $k - 1$, and for distinct positive integers m , and l ; we find that $f_m^2 f_m^{(k)}$ and $f_l^2 f_l^{(k)}$ share ω IM and \mathcal{F} is not normal at $z = 0$.

Also, $\omega \neq 0$ in Theorem 3.1 is essential. For example, let $\mathcal{F} = \{f_m\}$, where $f_m(z) = \frac{1}{e^{mz} + 1}$; $m = 1, 2, \dots$ and $z \in \mathbb{D}$. Choose $k = 2$, $n = 2$, $n_1 = 1$, and $n_2 = 0$, we have

$$M[f_m] = f_m^2 f'_m = -\frac{me^{mz}}{(e^{mz} + 1)^4} \neq 0.$$

Thus, for any $f, g \in \mathcal{F}$, $M[f]$ and $M[g]$ share 0 IM. But we see that \mathcal{F} is not normal in \mathbb{D} .

Theorem 3.2. *Let \mathcal{F} be a family of non-constant holomorphic functions on a domain D with all zeros of each $f \in \mathcal{F}$ having multiplicity at least k , where $k \geq 2$. Let $\omega \neq 0$ be a finite complex number and n_0, n_1, \dots, n_k be non-negative integers with $n_0 \geq 1$ and $n_1 + n_2 \dots + n_k \geq 1$. If there exists $\tilde{f} \in \mathcal{F}$ such that $M[\tilde{f}]$ share ω partially with $M[f]$ for every $f \in \mathcal{F}$, then \mathcal{F} is normal on D .*

As an illustration of Theorem 3.2, we have the following example:

Example 3.3. *Consider $\mathcal{F} = \{f_m(z) = me^{\frac{z}{m}} : m \in \mathbb{N}\}$, defined on \mathbb{C} . Take $k = 2, n = 1, n_1 = 0$, and $n_2 = 1$. Then*

$$M[f_m] = f_m f''_m = e^{\frac{2z}{m}},$$

and $M[f_m] = 1$ iff $\frac{2z}{m} = 2k\pi i, k \in \mathbb{Z}$ iff $z = mk\pi i$
For

$$m = 1; z = 0, \pm\pi i, \pm2\pi i, \pm3\pi i, \dots$$

$$m = 2; z = 0, \pm2\pi i, \pm4\pi i, \pm6\pi i, \dots$$

$$m = 3; z = 0, \pm3\pi i, \pm6\pi i, \pm9\pi i, \dots$$

and so on.

Thus for each $m \geq 2$, $M[f_m]$ share 1 partially with $M[f_1]$. Next, we have $\forall z, |z| \leq r, r > 0; |f_m(z)| = |me^{\frac{z}{m}}| = me^{\frac{\Re(z)}{m}} < me^{\frac{r}{m}} = M$, say, where $M > 0$ depends on r and this is true for each $m \in \mathbb{N}$. That is, \mathcal{F} is locally bounded on \mathbb{C} and hence by Montel Theorem \mathcal{F} is normal.

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A Value Distribution Result leading to Normality Criteria

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Abstract

In this article, we prove a distribution result for a certain class of differential polynomials and as a consequence prove a normality criterion concerning partially shared functions: Let \mathcal{F} be a family of meromorphic functions in a domain D . Let $m, k, n \geq k + 1$ be positive integers and $h \neq 0, \infty$ be a meromorphic function having no zeros and poles at the origin. If, there exists $f \in \mathcal{F}$ such that $f^m(f^n)^{(k)}$ share h partially with $\tilde{f}^m(\tilde{f}^n)^{(k)}$, $\forall f \in \mathcal{F}$, then \mathcal{F} is normal in D , provided $h \neq \tilde{f}^m(\tilde{f}^n)^{(k)}$.

1 Introduction and Main Results

For normal families of meromorphic functions, one may refer to [4]. Further, we define a small function of a meromorphic function f in $\mathbb{D}_R := \{z : |z| \leq R\}$ to be a meromorphic function ω satisfying

$$T(r, \omega) = o(T(r, f)) \text{ as } r \rightarrow R.$$

We say that f and g share a value $a \in \mathbb{C}$ IM if f and g have the same a -points counted with ignoring multiplicities. If multiplicities are counted, then they are said to share a CM (one may refer to [8]). In this paper, we use the idea of partial sharing of functions. A meromorphic function f is said to share a function ω partially with a meromorphic function g if

$$\overline{E}(\omega, f) \subseteq \overline{E}(\omega, g),$$

where $\overline{E}(\omega, \phi) = \{z \in \mathbb{C} : \phi(z) - \omega(z) = 0\}$, the set of zeros of $\phi - \omega$ counted with ignoring multiplicities.

In 1998, Y.Wang and M.Fang [7] proved:

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Theorem A: Let $k, n \geq k + 1$ be positive integers and f be a transcendental meromorphic function. Then $(f^n)^{(k)}$ assumes every finite non-zero value infinitely often.

In 2009, Yuntong Li and Yongxing Gu [3] gave the corresponding distribution result for rational functions:

Theorem B: Let $k, n \geq k + 2$ be positive integers, $a \neq 0$ be a finite complex number and f be a non-constant rational function. Then $(f^n)^{(k)} - a$ has at least two distinct zeros.

Corresponding to Theorem A and Theorem B, the normality criterion given by Yuntong Li and Yongxing Gu [3] is:

Theorem C: Let \mathcal{F} be a family of meromorphic functions in an arbitrary domain D . Let $k, n \geq k + 2$ be positive integers and $a \neq 0$ be a finite complex number. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share a in D for every pair of functions $f, g \in \mathcal{F}$, then \mathcal{F} is normal in D .

It is natural to ask whether Theorem A, Theorem B and Theorem C can be generalized for functions instead of constants and the sharing can be replaced by partial sharing. Yes, we have been able to answer these questions as an application of the following value distribution result for differential polynomials.

Theorem 1.1. *Let f be a transcendental meromorphic function and $m, k, n \geq k + 1$ be positive integers. Let*

$$F = f^m (f^n)^{(k)}.$$

Then

$$\left[\frac{k}{2(2k+2)} + o(1) \right] T(r, F) \leq \overline{N} \left(r, \frac{1}{F - \omega} \right) + S(r, F)$$

for any small function $\omega (\neq 0, \infty)$ of f .

Theorem 1.2. *Let $m, k, n \geq k + 1$ be positive integers and $\omega \neq 0$ be a finite complex number, and f be a non-constant rational function, then $f^m (f^n)^{(k)} - \omega$ has at least two distinct zeros.*

As an application of Theorem 1.1 and Theorem 1.2, we prove the following two normality criteria:

Theorem 1.3. *Let \mathcal{F} be a family of meromorphic functions in an arbitrary domain D . Let $m, k, n \geq k + 1$ be positive integers and $h \neq 0, \infty$ be a meromorphic function having no zeros and poles at the origin. If, there exists $\tilde{f} \in \mathcal{F}$ such that $f^m (f^n)^{(k)}$ share h partially with $\tilde{f}^m (\tilde{f}^n)^{(k)}$, $\forall f \in \mathcal{F}$, then \mathcal{F} is normal in D , provided $h \neq \tilde{f}^m (\tilde{f}^n)^{(k)}$.*

Remark: The condition $h \not\equiv \tilde{f}^m(\tilde{f}^n)^{(k)}$ can be omitted in Theorem 1.3 in case h is a small function of f on \mathbb{D}_R .

Theorem 1.4. *Let \mathcal{F} be a family of meromorphic functions in D . Let $m, k, n \geq k+1$ be positive integers and $h \not\equiv 0, \infty$ be a meromorphic function having no zeros and poles at the origin.*

If, for each $f \in \mathcal{F}$, $[f^m(f^n)^{(k)} - h](z) = 0$ implies $|(f^n)^{(k)}(z)| \leq A$, for some $A > 0$, then \mathcal{F} is normal in D .

2 Proof of Main Results

Proof of Theorem 1.1: Since F is a homogeneous differential polynomial in f of degree $n+m$, where exponents of f are positive integers, from [6], we have

$$T(r, f) + S(r, f) \leq CT(r, F) + S(r, F)$$

and

$$T(r, F) \leq BT(r, f) + S(r, f),$$

where B and C are constants, hence $T(r, \omega) = S(r, F)$ as $r \rightarrow \infty$. Therefore, ω is a small function of f iff ω is a small function of F .

Now, by Second Fundamental Theorem of Nevanlinna for three small functions (see [1] pp.47), we have

$$[1 + o(1)]T(r, F) \leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \omega}\right) + S(r, F) \quad (2.1)$$

Next, by using a result of Lahiri and Dewan (see [2], Lemma), we have

$$\begin{aligned} \overline{N}\left(r, \frac{1}{F}\right) &= \overline{N}\left(r, \frac{1}{f^m(f^n)^{(k)}}\right) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}_0\left(r, \frac{1}{(f^n)^{(k)}}\right) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + k\left[\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f)\right] + S(r, f) \\ &= (1+k)\overline{N}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f), \end{aligned} \quad (2.2)$$

where $\overline{N}_0\left(r, \frac{1}{(f^n)^{(k)}}\right)$ is the counting function ignoring multiplicity of those zeros of $(f^n)^{(k)}$ in $|z| \leq r$ which are not the zeros of f^n and hence f .

Also, if z_0 is a zero of f of order $p \leq k$, then z_0 is a zero of F of order $pn - k + mp \geq 2(n+m) - k \geq k+2+2m > k+3$ and if z_0 is a zero of f of order $p \geq k+1$, then z_0 is a zero of F of order $np - k + mp \geq (k+1)(m+n) - k \geq nk + (k+1)m + 1 > k(k+1) + 2$. Thus, it follows that

$$N\left(r, \frac{1}{F}\right) - \overline{N}\left(r, \frac{1}{F}\right) \geq (k+2)\overline{N}_k\left(r, \frac{1}{f}\right) + [k(k+1)+1]\overline{N}_{(k+1)}\left(r, \frac{1}{f}\right), \quad (2.3)$$

where $\overline{N}_{(k+1)}\left(r, \frac{1}{f}\right)$ and $\overline{N}_k\left(r, \frac{1}{f}\right)$ are the counting functions ignoring multiplicities of those zeros of f whose multiplicity is at least $k+1$ and at most k respectively.

From (2.2) and (2.3), we obtain

$$\begin{aligned}\overline{N}\left(r, \frac{1}{F}\right) &\leq (k+1)\overline{N}_{(k+1)}\left(r, \frac{1}{f}\right) + \frac{k+1}{k+2}\left[N\left(r, \frac{1}{F}\right) - \overline{N}\left(r, \frac{1}{F}\right)\right] \\ &\quad - \frac{k+1}{k+2}\left[(k\overline{k}+1+1)\overline{N}_{(k+1)}\left(r, \frac{1}{f}\right)\right] + k\overline{N}(r, f) + S(r, f) \\ &\leq N\left(r, \frac{1}{F}\right) - \overline{N}\left(r, \frac{1}{F}\right) + k\overline{N}(r, f) + S(r, f).\end{aligned}$$

That is,

$$\overline{N}\left(r, \frac{1}{F}\right) \leq \frac{1}{2}N\left(r, \frac{1}{F}\right) + \frac{k}{2}\overline{N}(r, f) + S(r, f).$$

Since $\overline{N}(r, F) = \overline{N}(r, f)$ and $S(r, F) = S(r, f)$, we have

$$\overline{N}\left(r, \frac{1}{F}\right) \leq \frac{1}{2}N\left(r, \frac{1}{F}\right) + \frac{k}{2}\overline{N}(r, F) + S(r, F).$$

Therefore (2.1) yields

$$[1+o(1)]T(r, F) \leq \frac{1}{2}N\left(r, \frac{1}{F}\right) + \frac{k+2}{2}\overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F-\omega}\right) + S(r, F). \quad (2.4)$$

Also, if z_0 is a pole of f of multiplicity p , then z_0 is a pole of F of multiplicity $np+k+mp \geq 2k+2$ and therefore,

$$N(r, F) \geq (2k+2)\overline{N}(r, F). \quad (2.5)$$

Finally, from (2.4) and (2.5), we find that

$$\left[\frac{k}{2(2k+2)} + o(1)\right]T(r, F) \leq \overline{N}\left(r, \frac{1}{F-\omega}\right) + S(r, F).$$

□

Proof of Theorem 1.2: If f is a polynomial, then $f^m(f^n)^{(k)}$ has at least one multiple zero, since $n \geq k+1$. By Fundamental Theorem of Algebra, $f^m(f^n)^{(k)} - \omega$ has at least one zero. Suppose $f^m(f^n)^{(k)} - \omega$ has exactly one zero, say z_0 . Then

$$f^m(f^n)^{(k)}(z) = \omega + A(z - z_0)^l, \text{ where } 0 \neq A \text{ is constant and } l > 0$$

Since $\omega \neq 0$, $f^m(f^n)^{(k)} - \omega$ has simple zeros only, which is not the case. Hence $f^m(f^n)^{(k)} - \omega$ has at least two distinct zeros.

Now, consider the case when f is rational but not polynomial. Suppose on the contrary that $f^m (f^n)^{(k)} - \omega$ has no distinct zeros. Then $f^m (f^n)^{(k)} - \omega$ has either exactly one zero or no zero.

First, we consider the case when $f^m (f^n)^{(k)} - \omega$ has exactly one zero. Let

$$f(z) = A \frac{\prod_{i=1}^s (z - \alpha_i)^{m_i}}{\prod_{h=1}^t (z - \beta_h)^{l_h}}, \quad (2.6)$$

where A is a non-zero constant, $m_i \geq 1 (i = 1, 2, \dots, s)$ and $l_h \geq 1 (h = 1, 2, \dots, t)$.

Put

$$M = \sum_{i=1}^s m_i \geq s \text{ and } N = \sum_{h=1}^t l_h \geq t. \quad (2.7)$$

Then

$$(f^n)^{(k)} = A^n \frac{\prod_{i=1}^s (z - \alpha_i)^{nm_i - k}}{\prod_{h=1}^t (z - \beta_h)^{nl_h + k}} g_k(z), \quad (2.8)$$

where

$$g_k(z) = n(M-N) [n(M-N) - 1] [n(M-N) - 2] \cdots [n(M-N) - k + 1] z^{k(s+t-1)} + \dots$$

is a polynomial of degree at most $k(s+t-1)$. Thus,

$$f^m (f^n)^{(k)} = A^{m+n} \frac{\prod_{i=1}^s (z - \alpha_i)^{(m+n)m_i - k}}{\prod_{h=1}^t (z - \beta_h)^{(m+n)l_h + k}} g_k(z) = \frac{P(z)}{Q(z)}, \text{ say.} \quad (2.9)$$

Since $f^m (f^n)^{(k)} - \omega$ has exactly one zero, z_0 say, from (2.9), we obtain

$$f^m (f^n)^{(k)} = \omega + \frac{B(z - z_0)^l}{\prod_{h=1}^t (z - \beta_h)^{(m+n)l_h + k}}, \quad (2.10)$$

where l is a positive integer and $B \neq 0$ is a constant.

Again, from (2.9), we have

$$\left(f^m (f^n)^{(k)} \right)' = A^{m+n} \frac{\prod_{i=1}^s (z - \alpha_i)^{(m+n)m_i - (k+1)}}{\prod_{h=1}^t (z - \beta_h)^{(m+n)l_h + (k+1)}} \tilde{g}(z), \quad (2.11)$$

where \tilde{g} is a polynomial with $\deg \tilde{g} \leq (k+1)(s+t-1)$.

Consequently (2.10), yields

$$\left(f^m (f^n)^{(k)} \right)' = A^{m+n} \frac{(z - z_0)^{l-1}}{\prod_{h=1}^t (z - \beta_h)^{(m+n)l_h + (k+1)}} \hat{g}(z), \quad (2.12)$$

where $\hat{g}(z) = l - [(m+n)N + kt] z^t + \dots$ is a polynomial.

Case-I: Suppose $l \neq (m+n)N + kt$. Then from (2.10) and using (2.9), we have

$$\begin{aligned}
& \deg P \geq \deg Q \\
& \Rightarrow \sum_{i=1}^s [(m+n)m_i - k] + \deg g_k \geq \sum_{h=1}^t [(m+n)l_h + k] \\
& \Rightarrow (m+n)M - ks + \deg g_k \geq (m+n)N + kt \\
& \Rightarrow (m+n)M - ks + k(s+t-1) \geq (m+n)N + kt \\
& \Rightarrow (m+n)N \leq (m+n)M - k < (m+n)M \\
& \text{i.e. } M > N.
\end{aligned}$$

Noting that $z_0 \neq \alpha_i; \forall i$, from (2.7), (2.11) and (2.12), we obtain

$$\begin{aligned}
& \sum_{i=1}^s [(m+n)m_i - (k+1)] \leq \deg \hat{g} = t \\
& \Rightarrow (m+n)M - (k+1)s \leq t \\
& \Rightarrow (m+n)M \leq (k+1)s + t \leq (k+1)M + N < (k+2)M \leq (m+n)M \\
& \text{i.e. } M < M, \text{ which is absurd.}
\end{aligned}$$

Case-II: Suppose $l = (m+n)N + kt$. It is sufficient to discuss the case $M \leq N$ here.

By comparing (2.11) and (2.12), we get

$$l - 1 \leq \deg \tilde{g} \leq (k+1)(s+t-1)$$

and hence

$$(m+n)N = l - kt \leq \deg \tilde{g} + 1 - kt \leq (k+1)(s+t-1) + 1 - kt \leq (k+2)N \leq (m+n)N$$

i.e. $N < N$, which is again absurd.

Finally, suppose $f^m (f^n)^{(k)} - \omega$ has no zero at all. Then $l = 0$ in (2.10), yields

$$f^m (f^n)^{(k)} = \omega + \frac{B}{\prod_{h=1}^t (z - \beta_h)^{(m+n)l_h + k}} \quad (2.13)$$

and so

$$\left(f^m (f^n)^{(k)} \right)' = \frac{BH(z)}{\prod_{h=1}^t (z - \beta_h)^{(m+n)l_h + (k+1)}}, \quad (2.14)$$

where $H(z)$ is a polynomial of degree $t - 1 < t$.

Proceeding as in the proof for Case-I, we again get a contradiction. This completes the proof. \square

Proof of the Theorem 1.3: Since normality is a local property, we may assume that $D = \mathbb{D}$. Suppose \mathcal{F} is not normal in \mathbb{D} . Then there exists at least one $z_0 \in \mathbb{D}$

such that \mathcal{F} is not normal at the point z_0 . W.l.o.g. we assume that $z_0 = 0$. By Zalcman's Lemma, there exists a sequence $\{f_j\}$ of functions in \mathcal{F} ; a sequence $\{z_j\}$ of complex numbers in \mathbb{D} with $z_j \rightarrow 0$ as $j \rightarrow \infty$; and a sequence $\{\rho_j\}$ of positive real numbers with $\rho_j \rightarrow 0$ as $j \rightarrow \infty$ such that the sequence $\{g_j\}$ of scaled functions

$$g_j(z) = \rho_j^{-\alpha} f_j(z_j + \rho_j z), \quad (2.15)$$

where $0 \leq \alpha < k$; converges locally uniformly to a non-constant meromorphic function $g(z)$ in \mathbb{C} with respect to the spherical metric. Moreover, $g(z)$ is of order at most 2.

Put $\alpha = \frac{k}{m+n} (< 1)$. Then

$$g_j^m(z) (g_j^n)^{(k)}(z) = f_j^m(z_j + \rho_j z) (f_j^n)^{(k)}(z_j + \rho_j z).$$

On every compact subset of \mathbb{C} that contains no poles of g , we get

$$\begin{aligned} \left[f_j^m (f_j^n)^{(k)} \right] (z_j + \rho_j z) - h(z_j + \rho_j z) &= \left[g_j^m (g_j^n)^{(k)} \right] (z) - h(z_j + \rho_j z) \\ &\rightarrow \left[g^m (g^n)^{(k)} \right] (z) - h(0) \\ &\rightarrow \left[g^m (g^n)^{(k)} \right] (z) - h_0, \end{aligned}$$

spherically uniformly, where $h_0 = h(0) \neq 0, \infty$.

Now, g is a non-constant meromorphic function and $g^m (g^n)^{(k)}$ is a homogeneous differential polynomial with exponents of g positive in each monomial. It follows that g and $g^m (g^n)^{(k)}$ have the same order and hence $g^m (g^n)^{(k)} \not\equiv h_0$. Thus, by Theorem 1.1 and Theorem 1.2, we find that $g^m (g^n)^{(k)} - h_0$ has at least two distinct zeros, say u_0 and v_0 . Since zeros are isolated, we can find two non-intersecting open disks $D(u_0, r)$ and $D(v_0, r)$ such that $D(u_0, r) \cup D(v_0, r)$ does not contain any zero of $g^m (g^n)^{(k)} - h_0$ different from u_0 and v_0 .

Thus, by Hurwitz's theorem, we see that, for sufficiently large values of j , there exist points $u_j \in D(u_0, r)$ and $v_j \in D(v_0, r)$ such that

$$\left[f_j^m (f_j^n)^{(k)} - h \right] (z_j + \rho_j u_j) = 0$$

and

$$\left[f_j^m (f_j^n)^{(k)} - h \right] (z_j + \rho_j v_j) = 0$$

Since by hypothesis $f_j^m (f_j^n)^{(k)}$ share h partially with $\tilde{f}^m (\tilde{f}^n)^{(k)}$, for some $\tilde{f} \in \mathcal{F}$, for every j , it follows that

$$\left[\tilde{f}^m (\tilde{f}^n)^{(k)} - h \right] (z_j + \rho_j u_j) = 0$$

and

$$\left[\tilde{f}^m (\tilde{f}^n)^{(k)} - h \right] (z_j + \rho_j v_j) = 0$$

Since $z_j + \rho_j u_j \rightarrow 0$ and $z_j + \rho_j v_j \rightarrow 0$ as $j \rightarrow \infty$, we find that

$$\left[\tilde{f}^m \left(\tilde{f}^n \right)^{(k)} - h \right] (0) = 0.$$

Since the zeros of $\left[\tilde{f}^m \left(\tilde{f}^n \right)^{(k)} - h \right]$ have no accumulation point, it follows that $z_j + \rho_j u_j = 0$ and $z_j + \rho_j v_j = 0$, for sufficiently large j , which is a contradiction to the fact that $D(u_0, r)$ and $D(v_0, r)$ are non-intersecting. \square

Proof of the Theorem 1.4: Proceeding as in the proof of the Theorem 1.3, we similarly find that $g^m (g^n)^{(k)} \neq h_0$. By Theorem 1.1 and Theorem 1.2, $g^m (g^n)^{(k)} - h_0$ must have a zero w_0 , say. By Hurwitz's Theorem, there is a sequence $w_j \rightarrow w_0$ as $j \rightarrow \infty$ such that

$$\left[f_j^m \left(f_j^n \right)^{(k)} - h \right] (z_j + \rho_j w_j) = 0$$

Again, since $\alpha = \frac{k}{m+n}$,

$$\begin{aligned} |(g_j^n)^{(k)}(w_j)| &= \rho_j^{k-\alpha n} |(f_j^n)^{(k)}(z_j + \rho_j w_j)| \\ &\leq A \rho_j^{k-\alpha n} \rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned}$$

Thus

$$\begin{aligned} (g^n)^{(k)}(w_0) &= \lim_{j \rightarrow \infty} (g_j^n)^{(k)}(w_j) = 0 \\ &\Rightarrow g^m (g^n)^{(k)}(w_0) = 0 \neq h_0, \end{aligned}$$

which is not possible. \square

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